Physics 798C
Superconductivity
Spring 2024
Homework 2
Due Thursday, 15 February, 2024

1. The Schrödinger equation for the Macroscopic Quantum Wavefunction $\Psi(\vec{r},t) = \sqrt{n^*(\vec{r},t)}\, e^{i\theta(\vec{r},t)}$ with n^* and θ real functions of position $\vec{r}$ and time t for a superconductor is $i\hbar \frac{\partial \Psi}{\partial t} = \frac{1}{2m^*}\left(-i\hbar\vec{\nabla} - q^*\vec{A}\right)^2 \Psi + q^* \phi \Psi$, where m^* and q^* are the (real) effective mass and charge of the superfluid charge carriers, and ϕ and $\vec{A}$ are the scalar and vector potentials, respectively.

a) Under the assumption that the (real) number density $n^*(\mathbf{r},t)$ is constant in space and time, derive the energy-phase relationship:

$$-\hbar\, \partial\theta/\partial t = (1/2n^*)\, \Lambda\, J_s^2 + q^*\, \phi$$

from the real part of the macroscopic quantum Schrödinger equation. Interpret this equation physically.

b) Now assume that $n^*(r,t)$ is NOT constant in either space or time. Show that the imaginary part of the macroscopic Schrödinger equation yields:

$$\partial n^*/\partial t = -\, \nabla \cdot (n^*\, \mathbf{v_s})$$

Interpret this result physically (it may help to multiply both sides by q^*).

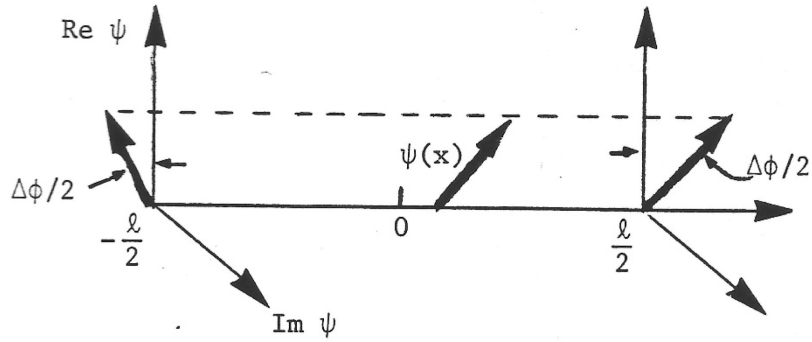
2. Fluxoid quantization can be written as $\oint_C \left(\Lambda \vec{J}_s + \vec{A}\right) \cdot d\vec{l} = n\Phi_0$, where n is any positive or negative integer or zero, and C is a circuit that runs entirely inside a superconductor. Using the expression for the supercurrent density in terms of the superfluid velocity, $\vec{J}_s = n^* q^* \vec{v}_s$, and the definition of the canonical momentum, $\vec{p}_{can} = m^* \vec{v}_s + q^* \vec{A}$, show that fluxoid quantization is an expression of the Bohr-Sommerfeld quantization condition: $\oint \vec{p} \cdot d\vec{q} = nh$, where (q,p) are conjugate coordinate and momentum, and h is Planck's constant.

3. A ring is made up of a film of thickness 1 μm, and has an inner diameter of 10 μm, and is formed of a superconducting material having London penetration depth $\lambda_L = 50$ nm at the temperature under consideration. Assume that the external magnetic field is zero and that the ring has trapped one flux quantum, Φ_0. Find the current density (in A/cm^2) at the inner surface of the ring. *Hint: Utilize fluxoid quantization and assume that the line integral of the vector potential from the trapped flux is zero along the inner surface of the ring.*

4. Here we discuss a model of a Superconductor / Normal Metal Barrier / Superconductor (SNS) Josephson junction. The wave function inside a short ($\ell \ll \xi_N$, where ξ_N is the normal metal coherence length) weak-link (normal metal barrier) Josephson junction is given by

$$\psi(x) = \psi_0 \left[\frac{-2x+\ell}{2\ell}\, e^{-i(\Delta\phi/2)} + \frac{2x+\ell}{2\ell}\, e^{+i(\Delta\phi/2)}\right], \text{ for } -\frac{\ell}{2} < x < +\frac{\ell}{2}, \text{ and } \psi_0 \text{ real.}$$

Note that $\psi\left(x = -\frac{\ell}{2}\right) = \psi_0 e^{-i(\Delta\phi/2)}$, and $\psi\left(x = +\frac{\ell}{2}\right) = \psi_0 e^{+i(\Delta\phi/2)}$, arising from a phase difference of $\Delta\phi$ across the junction. The complex wavefunction is shown in the Figure.

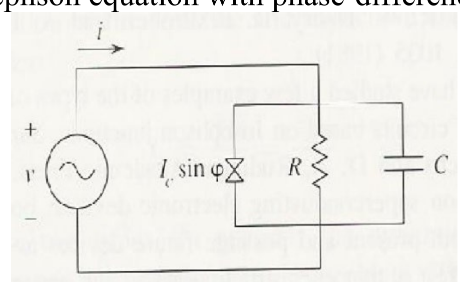


a) Show that the supercurrent-phase relationship for this junction is given by $J_s = \frac{e^* \hbar}{m^*} \frac{\psi_0^2}{\ell} \sin(\Delta\phi)$.

b) Show that the density of superconducting electrons in the barrier is $|\psi(x)|^2 = \psi_0^2 \left[\cos^2\left(\frac{\Delta\phi}{2}\right) + \left(\frac{2x}{\ell}\right)^2 \sin^2\left(\frac{\Delta\phi}{2}\right)\right]$, and the phase of the macroscopic quantum wavefunction is $\phi(x) = \tan^{-1}\left(\frac{2x}{\ell} \tan\left(\frac{\Delta\phi}{2}\right)\right)$, for $-\frac{\ell}{2} < x < +\frac{\ell}{2}$.

c) Sketch the density of superconducting electrons and the superfluid velocity $v_s = \hbar \nabla\phi / m^*$ as a function of position for $\Delta\phi = 0, \frac{\pi}{2}, \pi, 3\pi/2$.

5. Consider a constant voltage source with $v = V_0$ (DC voltage) connected across the generalized Josephson junction in the resistively and capacitively shunted junction (**RCSJ**) model as shown in the figure. The resistive channel is approximated by a constant resistance R, and the ideal Josephson junction with critical current I_c has a supercurrent given by the first Josephson equation with phase-difference φ.



(a) Show that the currents through the ideal junction, the resistor, and the capacitor are, respectively;

$$i_{JJ}(t) = I_c \sin\left(\frac{2\pi}{\Phi_0} V_0\, t + \varphi(0)\right),$$

$$i_R(t) = \frac{V_0}{R}, \text{ and}$$

$$i_C(t) = 0,$$

where $\varphi(0)$ is the initial phase difference across the ideal junction at $t = 0$.

(b) The total current $i(t)$ through the generalized junction is then $i(t) = i_{JJ} + i_R + i_C$. Show that the time-averaged current $\langle i(t) \rangle$ over one period of $\Theta = \Phi_0/V_0$ is $\langle i(t) \rangle = \frac{V_0}{R}$.